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# SOLUTION OF THE INTEGRAL EQUATIONS IN THE THREE-DIMENSIONAL NONSYMMETRICAL CONTACT PROBLEMS WITH THE FRICTION TAKEN INTO ACCOUNT

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Abstract. We develop a numerical-analytic method of solution of the integral equations with weak singularity for three-dimensional contact problems with complex multiply-connected domains. The proposed method is based on the potential expansion, a regularization of the first kind Fredholm equation that leads to the second kind equation and smoothing of the kernels. Simple layer potential expansion is developed when the density has no circular symmetry. This gives possibility to solve contact interaction problems for asymmetrical bodies and taking into account the friction and the roughness.

Keywords: contact problem, integral equation, kernel, regularization, potential expansion, small parameter, friction.

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## 1. Introduction

Computation of the potential type integrals arises in many theoretical and practical problems. Thus, stressed bodies weakened by flat cracks under a normal tear are characterized by the following integral-differential equation

$$p(\rho_0,\theta_0) = \Delta \iint_S \frac{\omega(\rho,\theta)}{r}\,\mathrm{d}s, \quad r^2 = \rho_0^2+\rho^2-2\rho_0\rho\cos(\theta_0-\theta), \;\; (\rho_0,\theta_0)\in s, \; \mathrm{d}s = \rho\mathrm{d}\rho\mathrm{d}\theta.$$

Signal decoding is realized by a solution of a potential problem in computer tomography and seismology. Mechanical dynamical systems have many parts with contact interaction. Boundary conditions on the surfaces of deformable bodies can be formulated in a way adequate to the reality only as a result of the contact problem solution [1]. Solutions of the contact problem get particular relevance in the development of tribology that studies contact of roughness surfaces taking into account losses on friction and wear, contact rigidity of movable and unmovable joints and cracks, etc. [2, 3, 8]. The main integral equation of the three-dimensional contact interaction problems serves to determine the vertical displacements $\delta$ and the normal pressure $p(x,y)$ under the punch neglecting the vertical displacement of the micro asperity resulting from the tangential force. The equation contains integrals with weak singularity of the simple layer potential type, in the general case taken over an unknown contact domain $s$ dependent on the friction coefficient [2]

$$\varphi_0(p(\rho_0,\theta_0)) + \iint_S \frac{\eta\, p(\rho,\theta)}{r}\,\mathrm{d}s + \iint_S \frac{\cos\widehat{rx}}{r}\,\psi_0(p(\rho,\theta))\,\mathrm{d}s = g(\rho_0,\theta_0)), \tag{1}$$

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where $g(\rho,\theta)$ depends on the punch contact surface, the indentation value and the angles of rotation; $\eta$ is a physical constant characterizing the material of the contacted bodies. The function $\varphi_0(p(\rho,\theta))$ characterizes the law of local deformation changes caused by the roughness of the surface of the elastic half-space (without the roughness $\varphi_0(p(\rho,\theta))=0$). The function $\psi_0(p(\rho,\theta))$ shows a dependence of the friction force and the normal pressure, taking into account the adhesion. For a contact interaction without the friction taken into account, one has $\psi_0(p(\rho,\theta))=0$.

An analysis of recent papers shows that earlier there were not found general solutions of the three dimensional contact problems for multiply-connected non-axe-symmetric domains taking into account the friction. This work proposes a solution of the problem using a simple layer potential expansion. The basis for such approach had been developed in the papers [5, 6, 8].

## 2. Simple layer potential expansion for an asymmetric density distribution

Let us consider a simple layer potential distributed on a flat annular ring when the density of the layer depends on the distance $\rho$ between the point and the center of the ring $\Omega$. Using the expansion of the generating function $P_k(z)$ into a series in the Legendre polynomials

$$\frac{1}{\sqrt{1+t^2-2t\cos\tau}}=\sum_{k=0}^{\infty}t^kP_k(\cos\tau),\quad t<1,$$

one gets the following expression for the simple layer potential

$$\begin{aligned}\iint_\Omega\frac{\sigma_0(\rho)}{r}\,\mathrm{d}\Omega &= 2\pi\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\times\\ &\times\left[\int_a^{\rho_0}\sigma_0(\rho)\left(\frac{\rho}{\rho_0}\right)^{2n+1}\mathrm{d}\rho+\int_{\rho_0}^{b}\sigma_0(\rho)\left(\frac{\rho_0}{\rho}\right)^{2n}\mathrm{d}\rho\right].\end{aligned}\tag{2}$$

The expression (2) coincides with the one obtained in [4].

Further on we consider the cases when the density of simple layer distribution is not axe-symmetric, but can be represented in the form

$$\sigma(\rho,\theta)=\sum_{n=0}^{\infty}\sigma_n(\rho)\cos(n\theta)+\bar{\sigma}_n(\rho)\sin(n\theta).$$

In the course of computation of the integral $\iint_\Omega\sigma_2(\rho)\cos2\theta/r\,\mathrm{d}\Omega$ with the help of the expansion of the generating function into a series in the Legendre polynomials one comes to the desire to compute the integral $\int_0^{2\pi}\cos2\theta\,P_n(\cos(\theta-\theta_0))\mathrm{d}\theta$.

One uses the expansion of the trigonometric functions into series in the Legendre polynomials, for example

$$\sin\tau=\tfrac{\pi}{4}\left[P_0(\cos\tau)-\textstyle\sum_{k=0}^{\infty}\frac{(2k+1)!!(2k-1)!!(4k+5)}{2^{2k+2}(k+2)!(k+1)!}P_{2k+2}(\cos\tau)\right]$$

and therefore

$$\iint_\Omega\sigma_2(\rho)\cos2\theta/r\,\mathrm{d}\Omega=\int_0^{2\pi}\cos2(\tau+\theta_0)\sum_{n=0}^{\infty}P_n(\cos\tau)U_n(\rho)\,\mathrm{d}\tau,$$

where

$$U_n(\rho)=\int_a^{\rho_0}\sigma_2(\rho)\left(\frac{\rho}{\rho_0}\right)^{n+1}\mathrm{d}\rho+\int_{\rho_0}^{b}\sigma_2(\rho)\left(\frac{\rho_0}{\rho}\right)^{n}\mathrm{d}\rho,\quad\theta-\theta_0=\tau.$$

The basis of the induction uses the Legendre polynomials properties:

$$A(1)=\iint_\Omega \sigma_2(\rho)\cos 2\theta/r\,\mathrm{d}\Omega=2\pi\cos 2\theta_0\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\frac{2n(2n+1)\cdot U_{2n}(\rho)}{(2n+2)(2n-1)}.$$

The second induction step is obtained in a similar way:

$$\begin{aligned} A(2)=\iint_\Omega \sigma_2(\rho)\cos 4\theta/r\,\mathrm{d}\Omega &= 2\pi\cos 4\theta_0\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\times \\ &\times \frac{2n(2n+1)(2n-2)(2n+3)}{(2n-3)(2n-1)(2n+2)(2n+4)}U_{2n}(\rho). \end{aligned}$$

Let us consider the induction parameter which equals to an even integer number $m$. Then it is possible to deduce the following inductive assumption

$$A(m)=\iint_\Omega \sigma_2(\rho)\cos 2m\theta/r\,\mathrm{d}\Omega=2\pi\cos 2m\theta_0\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 C_{m,n}\,U_{2n}(\rho), \tag{3}$$

where

$$C_{m,n}=\prod_{k=1}^{m}\left[\frac{(2n-2k+2)(2n+2k-1)}{(2n+2k)(2n-2k+1)}\right].$$

An equivalent form of the mathematical induction is used to prove this inductive assumption. Transformations similar to those made for $A(1)$ and $A(2)$ lead to the expression (3) for $A(m)$. The expression (3) can be used when the right hand side of the equation for $f(\rho,\theta)$ has a form of a series in sines and cosines of even multiples of $\theta$. The convergence is proved and it is also done on the boundary. Corresponding expressions are also found when $p(\rho,\theta)$ is given as a series in sines and cosines of odd multiples of $\theta$ [7].

## 3. Solution method of the problem on contact of a doubly-connected punch with a rough elastic half-space

A rigid cylindrical doubly-connected punch is indented by the vertical force $Q$ into a rough elastic half-space. The domain $s$ is the projection of the punch points contacting with the elastic half-space to the plane $z=0$. There is no loading out of the contact domain on the elastic half space. The boundary conditions are $p(\rho,\theta)=0, (\rho,\theta)\notin s$. One wants to find a solution of the system of equilibrium equations and of the main integral equation (1) which has the form [2]:

$$Bp(\rho_0,\theta_0)+\frac{1-\nu^2}{\pi E}\cdot\iint_S \frac{p(\rho,\theta)}{r}ds=g(\rho_0,\theta_0), \tag{4}$$

where $B$ is the factor characterizing deformation properties of roughness of the half-space surface, $\nu$ is the Poisson coefficient, $E$ is the modulus of elasticity,

$$s=\left\{(\rho,\theta):\ a\cdot(1+f(\varepsilon,\theta))\le\rho\le b\cdot(1+f(\varepsilon,\theta))\,,\ 0\le\theta\le 2\pi\right\},$$

$f(\varepsilon,\theta)$ is a continuously differentiable function which has a representation of the form

$$f(\varepsilon,\theta)=\sum_{i=1}^{\infty}\varepsilon^i f_i(\theta). \tag{5}$$

Assuming that the required distribution of the normal pressure $p(\rho,\theta)$ depends on $\varepsilon$, we suppose that the function $p(\rho,\theta)$ can be represented as a power series in the small parameter $\varepsilon$:

$$p(\rho,\theta)=\sum_{k=0}^{\infty}\varepsilon^k p_k(\rho,\theta). \tag{6}$$

To obtain the expansion of the potential from the equation (4) in a series in $\varepsilon$, one uses the following mapping of the domain $s$ onto the circular ring $\Omega$ [5, 6, 7]:

$$\rho = R\left(1+\sum_{i=1}^{\infty} f_i(\varphi)\varepsilon^i\right), \theta = \varphi\,, \tag{7}$$

$$\Omega = \{(R,\varphi) : a \le R \le b,\ \ 0 \le \varphi \le 2\pi\}\,, (\rho_0,\theta_0) \to (R_0,\varphi_0) \in \Omega.$$

In the new variables the potential density (6) can be written with the use of the transformation (7):

$$p(\rho(R,\varphi,\varepsilon),\varphi) = \sum_{i=0}^{\infty} P_i(R,\varphi)\varepsilon^i. \tag{8}$$

Taking into account that $R=\rho$ when $\varepsilon=0$, one gets the following expressions for the functions $P_i(R,\varphi)$ and represents here for $i=0,1,2$:

$$P_0(R,\varphi) = p_0(R,\varphi), P_1(R,\varphi) = p_1(R,\varphi) + p_0'(R,\varphi)\ R\ f_1(\varphi), \tag{9}$$

$$P_2(R,\varphi) = p_2(R,\varphi) + p_1'(R,\varphi)\ R\ f_1(\varphi) + p_0'(R,\varphi)\ R\ f_2(\varphi) +\ 1/2\cdot p_0''(R,\varphi)\ \ R^2\ f_1^2(\varphi).$$

Let us transform the integral from the equation (4) into the new variables. This is possible under the condition that the mapping (7) of the domain $s$ onto the circular ring $\Omega$ is one-to-one and continuously differentiable. The ratio of the image measure with the original one equals the Jacobian of the transformation. The singular point is cut out by the circle of a small radius $\alpha$. Now the domain $s$ with the cut out point maps to the domain $\Omega-\alpha$ with the boundary dependent on $\varepsilon$. Since the integrand and the equations of the domain contour depend on $\varepsilon$, we differentiate the equation taking into account dependence of the boundary equation on the parameter to obtain an expansion of the integral with weak singularity. Assuming, that $p_i(\rho,\theta)$ and $f_i(\theta)$ are continuously differentiable functions inside the domain $s$, after the limit transformation $\alpha \longrightarrow 0$, we obtain the simple layer potential expansion in the new variables [7]:

$$\iint_S \frac{p(\rho,\theta)}{r} ds = \sum_{i=0}^{\infty} \varepsilon^i \left[\iint_\Omega \frac{P_i(R,\varphi)}{r(R,R_0)} d\Omega + H_i(P_0,P_1,...,P_{i-1})\right], \tag{10}$$

where for $i=0,1,2: H_0=0, \qquad H_1(P_0)=D_1(P_0), \qquad H_2(P_0,P_1)=D_1(P_1)+D_2(P_0)$,

$$D_1(P_0) = \iint_\Omega \frac{P_0(R)}{r(R,R_0)} f_1(\varphi)\mathrm{d}\Omega - R_0\frac{\partial}{\partial R_0}\iint_\Omega \frac{P_0(R)}{r(R,R_0)}[f_1(\varphi)-f_1(\varphi_0)]\mathrm{d}\Omega,$$

$$\begin{aligned} D_2(P_0) = \iint_\Omega \frac{P_0(R)}{r(R,R_0)} f_2(\varphi)\mathrm{d}\Omega - R_0\frac{\partial}{\partial R_0}\iint_\Omega \frac{P_0(R)}{r(R,R_0)}[f_2(\varphi)-f_2(\varphi_0)]\mathrm{d}\Omega +\\ + R_0\frac{\partial^2}{\partial R_0^2}\iint_\Omega \frac{P_0(R)}{2r(R,R_0)}[f_1(\varphi)-f_1(\varphi_0)]^2\mathrm{d}\Omega, \end{aligned} \tag{11}$$

$$r^2(R,R_0) = R^2 + R_0^2 - 2R\cdot R_0\cos(\varphi-\varphi_0);\ \ (R_0,\varphi_0)\in\Omega,$$

$\Omega = \{(R,\varphi) : a \le R \le b,\ \ 0 \le \varphi \le 2\pi\}$. The coefficients of different powers of $\varepsilon$ in (10) depend on the similar potentials distributed on the circular ring domain. Then, if the desired functions are Lipschitzian with positive index $P_i(\rho,\theta) \in Lip_\alpha$ $(\alpha > 0)$, the integrals are differentiable necessary number of times. The doubly-connected domain $s$ is bounded by the lines $\Gamma_1, \Gamma_2$ with the following equations

$$\rho_{\Gamma 1} = a\ \ (1+f(\varepsilon,\theta))\,, \rho_{\Gamma 2} = b\ \ (1+f(\varepsilon,\theta))\,, \tag{12}$$

where one has the expression (5) for $f(\varepsilon,\theta)$.

The right hand side part of the main integral equation of the problem (4) has the form [2] $g(\rho,\theta) = \delta + \beta_1\rho\cos\theta - \beta_2\rho\sin\theta + f_0(\rho,\theta)$. Here $\beta_1, \beta_2$ are the rotation vector projections, $\delta$ is the value of the punch indentation; they are determined from the punch equilibrium equations. $f_0(\rho,\theta)$ is a function of the punch base surface, for the flat punch one has $f_0(\rho,\theta) = 0$ . It is natural to assume that, unknown $\delta$, $\beta_1, \beta_2$ depend on the parameter $\varepsilon$. This parameter characterizes the equation of the boundaries of the contact domain $s$ (12). So they can be written as the power series in $\varepsilon$

$$\delta = \sum_{k=0}^{\infty} \delta_k \varepsilon^k; \beta_1 = \sum_{k=0}^{\infty} \beta_{1k}\varepsilon^k; \beta_2 = \sum_{k=0}^{\infty} \beta_{2k}\varepsilon^k; g(\rho,\theta) = \sum_{k=0}^{\infty} g_k(\rho,\theta)\, \varepsilon^k. \tag{13}$$

The integrals in the equilibrium equations can be represented as power series in $\varepsilon$. We transform the main integral equation (4) of the problem and the equilibrium equations into similar recurrent systems to determine $P_k(R,\varphi), \delta_k, \beta_{1k}, \beta_{2k}$ , where the integrals are distributed only on the circular ring. Let us present the recurrent system to determine $P_1(R,\varphi)$. After application of the potential expansion at an inner point (10), the equation (4) is reduced to the following system . One uses the potential expansion with the asymmetric density (3) in the asymmetrical case.

$$BP_k(R_0,\varphi_0) + \frac{1-v^2}{\pi E}\left[\iint_\Omega \frac{P_k(R,\varphi)}{r(R,R_0)} d\Omega + H_k\left(P_0, P_1, ..., P_{k-1}\right)\right] = G_k(R_0,\varphi_0), \tag{14}$$

where the functions $G_k(R,\varphi)$ are defined by $g_k(R,\varphi)$ with dependencies similar to (9) $H_k(P_0, P_1, ..., P_{k-1})$ using expressions (11), $\mathrm{d}\Omega = R\mathrm{d}R\mathrm{d}\varphi, k = 0, 1, ....$ We reduce the problem for the domain close to the circular ring $s$ bounded by (12) to a recurrent sequence of similar problems for the domain $\Omega$ bounded by the circles $R = a$ and $R = b$.

Let us consider the contact doubly-connected domain bounded by the Booth lemniscates, which represent deformed ellipses. The equations of such boundaries in the polar coordinate system are

$$\rho_1(\theta) = a\ \left(1 - \varepsilon^2\sin^2\theta\right)^{1/2}; \quad \rho_2(\theta) = b\ \left(1 - \varepsilon^2\sin^2\theta\right)^{1/2}, \tag{15}$$

where $\varepsilon$ is the eccentricity of the ellipse deformed into Booth lemniscates, $\varepsilon^2 = 1 - a_1^2/a^2 = 1 - b_1^2/b^2, a, b$ are focal parameters and $a_1, b_1$ are small semiaxes. The equation of the lines bounding the contact domain (15) can be written in the form of expansions in the power series (7), so the functions $f_i(\varepsilon,\theta)$, $i = 1, 2$, in the expansion (5) have the form:

$$f_1(\theta) = 1/4 \cdot (\cos 2\theta - 1)\,, \quad f_2(\theta) = 1/64 \cdot (-\cos 4\theta + 4\cos 2\theta - 3)\,. \tag{16}$$

The following system of two-dimensional integral equations is obtained as the first approximation of the equations (14) when $k = 0$, taking into account the equilibrium equations,

$$Bp_0(\rho_0,\theta_0) + \frac{(1-\nu^2)}{(\pi E)} \cdot \iint_\Omega \frac{p_0(\rho,\theta)}{r}\, d\Omega = \delta_0. \tag{17}$$

$$\iint_\Omega p_0(\rho,\theta)\ \mathrm{d}\Omega = Q.$$

After introducing the notations [2, 8]:

$$B\pi E/\left(b\cdot(1-\nu^2)\right) = B_1; \left(1-\nu^2\right)/(\pi E)\cdot p_0(\rho)/\delta_0 = \varphi(\rho), \tag{18}$$

we transform the equation (17) to the one-dimensional integral equation

$$\frac{B_1}{2\pi}\varphi(\rho_0) + \int_a^b \mathrm{K}(\rho_0,\rho)\cdot\varphi(\rho)\ \frac{d\rho}{b} = \frac{1}{2\pi b}, \tag{19}$$

where the kernel of the integral equation has the form:

$$\mathrm{K}(\rho_0,\rho)=\begin{cases}\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\left(\frac{\rho}{\rho_0}\right)^{2n+1}, & \rho<\rho_0;\\ \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\left(\frac{\rho_0}{\rho}\right)^{2n}, & \rho>\rho_0.\end{cases}\tag{20}$$

We represent the integral equation (19) in the equivalent form as in the paper [8]

$$\frac{B_1}{2\pi}\varphi(\rho_0)+\int_a^b \mathrm{K}(\rho_0,\rho)\cdot(\varphi(\rho)-\varphi(\rho_0))\ \frac{d\rho}{b}+\varphi(\rho_0)\int_a^b \mathrm{K}(\rho_0,\rho)\cdot\ \frac{d\rho}{b}=\frac{1}{2\pi b},\tag{21}$$

where the integrand in the first integral is well defined, because the difference $(\varphi(\rho)-\varphi(\rho_0))$ is equal to zero on the diagonal when $\rho=\rho_0$. The computation of the integral $\int_a^b \frac{\mathrm{K}(\rho_0,\rho)}{b}d\rho$ is made without the desired functions and can be represented in an explicit form:

$$\int_a^b \mathrm{K}(\rho_0,\rho)\frac{d\rho}{b}=\frac{2}{\pi}\left[E\left(\frac{\rho_0}{b}\right)+\frac{\rho_0}{b}\left(\left(1-\frac{a^2}{\rho_0^2}\right)\cdot \mathrm{K}\left(\frac{a}{\rho_0}\right)-E\left(\frac{a}{\rho_0}\right)\right)\right].\tag{22}$$

Here $K(z),E(z)$ are the elliptic integrals of the first and the second kind correspondingly.

If the equation (21) is multiplied by $\lambda=2\pi/B_1$, we obtain the standard form of the Fredholm second kind equation with weak singularity. The Fredholm operator with weak singularity is completely continuous, and therefore, the operator $I+A$ has an inverse limited operator for $\parallel A\parallel\leq q<1$ ($I$ is the unit operator). The estimates for the values of $\lambda$ are known from the Fredholm theory. Using the expressions for the kernel (20) and (22), they can be obtained as $\lambda<1/(4bK(a/b))$.

The solution of the problem is also important for smooth surfaces. In this case the main equation (4) is of the first kind [7]. So, when $B_1$ is close to zero, the approximate solution of the first kind is obtained, which corresponds to the absence of the roughness $B_1=0$ in this problem.

Solution of the second kind Fredholm integral equations can be written as a series for the resolvent. This series represents an expansion in power series in the parameter $\lambda$ near the point $\lambda=0$, and therefore converges till the first singular point $\lambda_1$ of this function, that is the first eigenvalue of the kernel. Consequently, the series for the resolvent can not be used if $\mid\lambda\mid>\mid\lambda_1\mid$, since it diverges there. It is difficult to use the series if $\mid\lambda\mid$ is close to $\mid\lambda_1\mid$, since then it converges slowly. N. Bogolyubov and N. Krylov recommended to apply the analytic continuation. In this case, the simplest variant is to substitute it for $B_1/(2\pi)=1-\alpha$, where $0<\alpha<1$ . When $\alpha=1$, the equation of the second kind (17) is transformed into the equation of the first kind [8]. Taking into account the substitution, the equation (17) has the following form

$$(1-\alpha)\varphi(\rho_0)+\int_a^b \mathrm{K}(\rho_0,\rho)\cdot(\varphi(\rho)-\varphi(\rho_0))\ \frac{d\rho}{b}+\varphi(\rho_0)\int_a^b \mathrm{K}(\rho_0,\rho)\cdot\ \frac{d\rho}{b}=\frac{1}{2\pi b}.\tag{23}$$

This equation can be solved by successive approximations when $\alpha<1$. The unknown constant $\delta_0$ is determined from the equilibrium equations. When $k=1$, taking into account the equilibrium equations, we obtain the system of the two-dimensional integral equations at the second approximation of the equations (14):

$$Bp_1(\rho_0,\theta_0)+\frac{(1-\nu^2)}{\pi E}\times$$
$$\times\left[\iint_\Omega \frac{P_1(\rho,\theta)}{r}\,d\Omega+\left(1-\rho_0\frac{\partial}{\partial\rho_0}\right)\iint_\Omega\left(\frac{-P_0(\rho)}{4\cdot r}+\frac{P_0(\rho)}{4\cdot r}\cos 2\theta\right)d\Omega\right]=\delta_1.\tag{24}$$

$$\iint_{\Omega} P_1(\rho,\theta)\, d\Omega - 1/2 \iint_{\Omega} P_0(\rho,\theta)\, d\Omega = 0.$$

After transformation, the substitution $P_1(\rho,\theta) = p_{10}(\rho) + p_{12}(\rho) cos(2\theta)$ and taking into account the functions of $P_k(\rho,\theta)$ and $p_k(\rho,\theta)$, the first equation of (24) is transformed into the following two equations

$$Bp_{10}(\rho_0) + \frac{(1-\nu^2)}{\pi E} \cdot \iint_{\Omega} \frac{p_{10}(\rho,\theta)}{r}\, d\Omega = \delta_1 + \frac{1}{4}\delta_0 - \frac{1}{4} B P_0(\rho_0), \tag{25}$$

$$\begin{aligned} &Bp_{12}(\rho)\cos 2\theta_0 + \frac{(1-\nu^2)}{\pi E} \cdot \iint_{\Omega} p_{12}(\rho,\theta)\cos 2\theta / r\, d\Omega = \\ &= \frac{1}{4}\rho_0 B\, P_0'(\rho)\cos 2\theta_0 - \frac{(1-\nu^2)}{\pi E} \cdot \left(1 - \rho_0 \frac{\partial}{\partial \rho_0}\right) \cdot \iint_{\Omega} \frac{P_0(\rho)}{4 \cdot r} \cos 2\theta\, d\Omega. \end{aligned} \tag{26}$$

On the base of the form of the equation (23), we find out

$$p_{10} = \delta_1/\delta_0 \cdot p_0(\rho) + 1/4 p_0(\rho) + \tilde{p}_{10}(\rho).$$

After the substitution $\frac{1-\nu^2}{\pi E}\frac{\tilde{p}_{10}(\rho)}{\delta_0} = \psi(\rho)$, $\frac{B_1}{2\pi} = 1 - \alpha$, the following equation is obtained

$$\begin{aligned} &(1-\alpha)\cdot\psi(\rho_0) + \int_a^b \mathrm{K}(\rho_0,\rho)\cdot(\psi(\rho) - \psi(\rho_0))/\mathrm{b}\cdot\, d\rho = \\ &= -0,25\cdot\varphi(\rho_0) - \psi(\rho_0)\int_a^b \mathrm{K}(\rho_0,\rho)/\mathrm{b}\cdot\, d\rho. \end{aligned} \tag{27}$$

The kernel of this equation has the same form as in the equation (23) and it is also solved by the method of successive approximations. The indentation value $\delta_1$ is determined from the second equation in the system (24). We make similar transformations, use the developed computing methods of the simple layer potential with asymmetric density distribution of the potential [7] and obtain that the integral equation (26) is transformed into

$$\begin{aligned} u(\rho_0) = \alpha\, u(\rho_0) + \frac{1}{4}(1-\alpha)\cdot\varphi(\rho_0) - \frac{1}{8\pi b} - \int_a^b \mathrm{K}_1(\rho_0,\rho)\cdot(u(\rho) - u(\rho_0))\ \frac{d\rho}{b} - \\ - \frac{u(\rho_0)}{b}\cdot\int_a^b \mathrm{K}_1((\rho_0,\rho))\, d\rho - \frac{1}{4b}\cdot\int_a^b \mathrm{K}_2(\rho_0,\rho)\,\varphi(\rho)\cdot\ d\rho, \end{aligned} \tag{28}$$

where

$$u(\rho) = p_{12}(\rho)\cdot\frac{(1-\nu^2)}{\delta_0 \pi E}, \tag{29}$$

$$\mathrm{K}_1(\rho_0,\rho) = \begin{cases} \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \left[1 + \frac{2}{(2n+2)(2n-1)}\right]\left(\frac{\rho}{\rho_0}\right)^{2n+1}, & \rho < \rho_0; \\ \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \left[1 + \frac{2}{(2n+2)(2n-1)}\right]\left(\frac{\rho_0}{\rho}\right)^{2n}, & \rho > \rho_0. \end{cases}$$

$$\mathrm{K}_2(\rho_0,\rho) = \begin{cases} \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{2}{(2n-1)}\left(\frac{\rho}{\rho_0}\right)^{2n+1}, & \rho < \rho_0; \\ \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{2}{(2n+2)}\left(\frac{\rho_0}{\rho}\right)^{2n}, & \rho > \rho_0. \end{cases}$$

A similar transformation is done in the third approximation. As a result, we obtain the solution of the problem of indentation into the rough elastic half-space of a flat punch with the base in the form close to a circular ring bounded by lines with the equation of the Booth lemniscates, taking into account the first three approximations.

## 4. Solution method of the problem on contact of doubly-connected punch taking into account the friction

A rigid punch is pressed by the vertical force $Q$ into the elastic half-space. There is also the moving force $T$ applied on the height $d$ from the punch base, which is balanced by the friction force. Coulomb's linear friction law [2] is considered here. Let us consider the moving force which is directed parallel to the axis $Ox$. The main integral equation of the problem without taking into account the roughness (1) is the first kind Fredholm equation [2]:

$$\frac{1-\nu^2}{\pi E}\iint_\Omega \frac{p(\rho,\theta)}{r}\left[1+\varepsilon_1\ \cos r\hat{\ }x\right]\ d\Omega=\delta, \tag{30}$$

where $\varepsilon_1=\mu\cdot(1-2\nu)/(2-2\nu), \mu$ is the coefficient of friction. Let us assume that $\varepsilon_1$ is a small parameter as it is much smaller than 1 in the most cases.

Regularization of the equation (30) leads to the solution of the second kind equation, when $\alpha<1, P(\rho,\theta)=\frac{1-\nu^2}{\pi E}\cdot p((\rho,\theta),$ (23):

$$(1-\alpha)\cdot P\left(\rho,\theta\right)+\iint_\Omega \frac{P}{2\pi br}\cdot\left[1+\varepsilon_1\frac{\rho_0\cos\theta_0-\rho\cos\theta}{r}\right]\ dS=\frac{\delta}{2\pi b}, \tag{31}$$

The equation (31) can be considered as the basic integral equation of the problem of the punch indentation into the elastic half-space with the roughness and the friction. In this case we neglect the vertical displacement of micro asperities due to the action of the tangent force. To compute the integrals in the equation (31) we use their expansions (3), (10) [7, 8].

$$\iint_\Omega \frac{\cos rx}{r}P(\rho,\theta)d\Omega=\pi\sum_{n=1}^{\infty}\int_a^{\rho_0}\left[a_n\left(\rho\right)\cos\left(n+1\right)\theta_0+b_n\left(\rho\right)\sin\left(n+1\right)\theta_0\right]\cdot\left(\frac{\rho}{\rho_0}\right)^{n+1}d\rho-$$

$$-\pi\sum_{n=1}^{\infty}\int_{\rho_0}^{b}\left[a_n\left(\rho\right)\cos\left(n-1\right)\theta_0+b_n\left(\rho\right)\sin\left(n-1\right)\theta_0\right]\cdot\left(\frac{\rho_0}{\rho}\right)^{n-1}d\rho+\pi\cos\theta_0\int_a^{\rho_0}a_n(\rho)\cdot\left(\frac{\rho}{\rho_0}\right)d\rho,$$

where $P\left(\rho,\theta\right)=a_0\left(\rho\right)/2+\sum_{n=1}^{\infty}a_n\left(\rho\right)\cos n\theta+b_n\left(\rho\right)\sin\theta$.

The zero approximation $p_0(\rho,\theta)$ is the solution of the integral equation for $\varepsilon_1=0$. We consider the solution $P(\rho,\theta)$ of the equation (30), close to $p_0(\rho,\theta)$ when the parameter $\varepsilon_1$ is small. In this case, let us represent the desired function and the indentation value as the following series

$$P\left(\rho,\theta\right)=\sum_{i=0}^{\infty}\varepsilon_1^i p_i\left(\rho,\theta\right);\delta=\sum_{i=0}^{\infty}\varepsilon_1^i\delta_i. \tag{32}$$

The equations to determine the expansion coefficients $P(\rho,\theta)$ in power series in $\varepsilon_1$ for three approximations are

$$(1-\alpha)\,p_0\left(\rho,\theta\right)+\iint_\Omega \frac{p_0\left(\rho,\theta\right)}{2\pi br}d\Omega=\frac{\delta_0}{2\pi b}, \tag{33}$$

$$(1-\alpha)\,p_1\left(\rho,\theta\right)+\iint_\Omega \frac{p_1\left(\rho,\theta\right)}{2\pi br}\ d\Omega+\iint_\Omega p_0\left(\rho\right)\ \frac{\rho_0\cos\theta_0-\rho\cos\theta}{2\pi br^2}\ d\Omega=\frac{\delta_1}{2\pi b},$$

$$(1-\alpha)\,p_2\left(\rho,\theta\right)+\iint_\Omega \frac{p_2\left(\rho,\theta\right)}{2\pi br}\ d\Omega+\iint_\Omega p_1\left(\rho,\theta\right)\ \frac{\rho_0\cos\theta_0-\rho\cos\theta}{2\pi br^2}\ d\Omega=\frac{\delta_2}{2\pi b}.$$

Without the roughness $\alpha=1$, the equations (31), (33) are of the first kind, and the exact solutions can be obtained in each approximation. For example, the first two approximations of the pressure distribution under a flat circular punch, and under the parabolic punch taking into account the friction and the adhesion are obtained in [8]. We transform the equations (31), (33) into a one-dimensional to apply the method of successive approximations. And the equation (23) is used for the determination of $p_0(\rho,\theta)$. The following equation (similar to the one above) is obtained for the determination of $p_1(\rho,\theta) = g(\rho)cos\theta$:

$$g(\rho_0) = \alpha g(\rho_0) - \int_a^b \frac{K_{11}(\rho_0,\rho)\left[g(\rho) - g(\rho_0)\right]}{b} d\rho - \frac{g(\rho_0)}{b}\int_a^b K_{11}(\rho_0,\rho)\, d\rho - \int_a^{\rho_0} \frac{\rho\, g(\rho)}{b\cdot\rho_0} d\rho,$$

$$K_{11}(\rho_0,\rho) = \begin{cases} \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \left[1 - \frac{1}{2n+2}\right] \left(\frac{\rho}{\rho_0}\right)^{2n+2}, & \rho < \rho_0; \\ \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \left[1 - \frac{1}{2n+2}\right] \left(\frac{\rho_0}{\rho}\right)^{2n+1}, & \rho > \rho_0. \end{cases}$$

While applying approximate and regularization methods, the error of the change of an integral operator by a discrete one effects the result. N. Bogolyubov and N. Krylov developed an effective method of changing an integral equation system by algebraic equations, using the average values of the unknown function at each partition of the integration domain. The introduction of the difference between the values of the unknown function at different points and the subsequent interpolation of the summands with the same indexes of the points are suggested to resolve the singularities. We obtain the recurrent expression using one of the cubature formulas for the numerical solution of the problem taking into account the friction and the roughness for an arbitrary unknown contact domain [8, 9].

## 5. Numerical results

The surface of pressure distribution $p/p*, p* = Q/(2\pi b^2)$ under the punch bounded by the lines of the Booth lemniscates, $\varepsilon = 0,65$, is given in fig. 1, values of the dimensionless parameters are $\varepsilon_1 = 0,057; d/b = 0,7; \alpha = 0,8; a/b = 0,35$. The lines of equal pressure under the surface show substantial asymmetry in the distribution of the normal pressure under the punch, which increases with the value of its eccentricity.

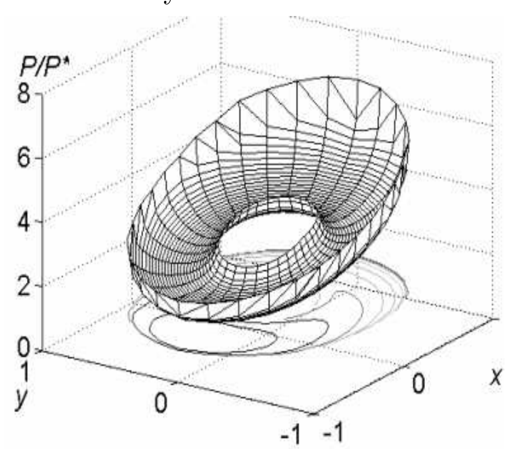


Figure 1. The normal pressure distribution under the punch with the base bounded by the Booth lemniscates

The lines of equal pressure are closed near the points with the minimum pressure. They take the form of curves, similar to the contours of the contact domain, closer to the boundaries. Increasing the application height of the horizontal force leads to a greater asymmetry of the pressure distribution that can result in lifting the punch from the surface of the elastic half-space. A zone of the negative pressure appears at the value of $d/b = 1,6$.

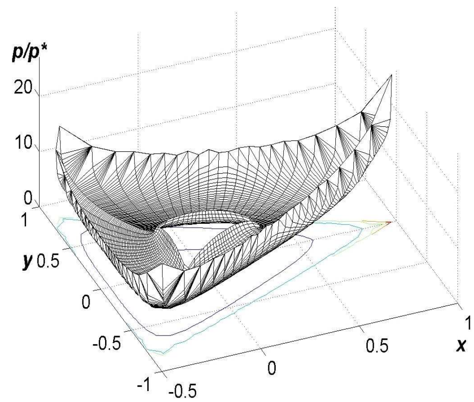


Figure 2. The normal pressure distribution under the punch with the base bounded by the triangles

The surface of the pressure distribution is shown in fig. 2. Corresponding curves of equal pressure are in fig. 3. The contact domain is bounded by the triangles. The linear friction law is taken into account. The dimensionless parameters are $a/b = 0,3; \varepsilon_1 = 0,057; d/b = 0,35$ and $\alpha = 0,99$.

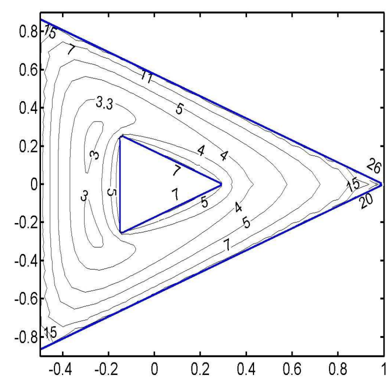


Figure 3. Contours of equal pressure under the punch with the base bounded by the triangles

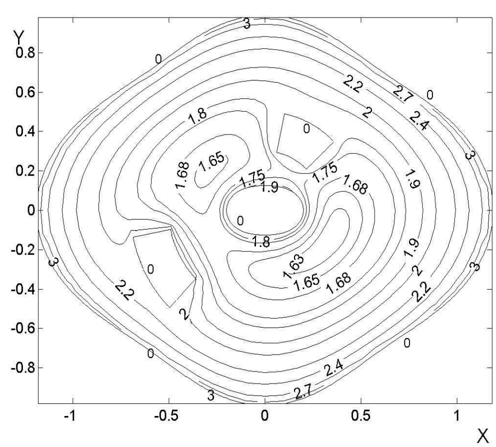


Figure 4. Contours of equal pressure under the punch with the multiply-connected base

The lines of equal pressure are concentrated near the boundaries of the contact domain. The roughness leads to the normal pressure taking the maximum finite values at the boundaries of the contact domain of the flat punches. Whereas in the case of the completely smooth contact, the normal pressure tends to infinity at the boundaries of the flat punches.

The multiply-connected contact domain of the flat punch is close to the rhomb with three holes: two in the shape of truncated sectors, and one elliptical at the center. The equal pressure contours are represented in fig. 4 for $\alpha = 0,7; \varepsilon_1 = 0,057; d/b = 0,4$ .

The numerical examples show that with the increase of the half-space roughness the maximum of the normal pressure is decreased and the minimum of the pressure is increased. This leads to a more uniform distribution of the pressure on the contact domain.

## 6. Conclusions

We have developed a solution method of the integral equations with weak singularity for contact problems with complex multiply-connected domains taking into account the roughness and the friction. We have obtained a simple layer potential expansion when the density has no circular symmetry. The proof has been made by the mathematical induction. The expansion convergence has been shown and also done at the boundaries.

The proposed numerical-analytic method is based on the potential expansion, the regularization of the first kind Fredholm equation that leads to the second kind one and smoothing the kernels. Then the system of integral equations can be solved by numerical methods. Successive approximations are used here.

We have developed the numerical-analytic solution for the flat punch with the doubly-connected base in the form of the Booth lemniscates. The normal pressure distributions, the rotations angles and the indentation values have been obtained and compared for different contact domains taking into account the linear roughness and the friction. We have studied the stability of the punch depending on the height of application of the moving force, the friction coefficient and the punch shape. The roughness leads to the fact that the normal pressure takes the maximum finite values at the boundaries of the contact domain of the flat punches, whereas in the case of completely smooth contact, the normal pressure tends to infinity at the boundaries of the flat punches.

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